

On simultaneous approximation of the values of certain Mahler functions

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Abstract

In this paper, we estimate the simultaneous approximation exponents of the values of certain Mahler functions. For this we construct Hermite-Padé approximations of the functions under consideration, then apply the functional equations to get an infinite sequence of approximations and use the numerical approximations obtained from this sequence.

1 Introduction

Let $\alpha_1, \dots, \alpha_n$ be real numbers. The *simultaneous approximation exponent* $\mu(\alpha_1, \dots, \alpha_n)$ of $\alpha_1, \dots, \alpha_n$ is the supremum of the real numbers μ such that the inequality

$$\max_{1 \leq i \leq n} \left| \alpha_i - \frac{p_i}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p_i/q . If at least one of α_i is irrational, then $\mu \geq 1 + 1/n$, see e.g. W. Schmidt [14]. In the case $n = 1$, $\mu(\alpha_1)$ is called the irrationality exponent of α_1 . Recently a remarkable progress has been achieved in proving that $\mu(\alpha_1) = 2$ for many classes of so-called automatic numbers and more generally the values of Mahler functions, see in particular [4, 5, 9, 11, 15, 18] and the references there in. In the present work our purpose is to study the simultaneous approximation exponents $\mu(\alpha_1, \alpha_2)$ for some numbers α_1 and α_2 of the above mentioned type.

Our first result considers generating functions of Stern's sequence $(a_n)_{n \geq 0}$ and its twisted version $(b_n)_{n \geq 0}$ defined by the recursions

$$\begin{cases} a_0 = 0, & a_1 = 1, \\ a_{2n} = a_n, & a_{2n+1} = a_n + a_{n+1}, \end{cases} \quad (n \geq 1), \quad (1.1)$$

and

$$\begin{cases} b_0 = 0, & b_1 = 1, \\ b_{2n} = -b_n, & b_{2n+1} = -(b_n + b_{n+1}), \end{cases} \quad (n \geq 1). \quad (1.2)$$

It is proved in [5, Theorem 2.4] that $\mu(A(1/b)) = \mu(B(1/b)) = 2$ for all integers $b \geq 2$, where

$$A(z) = \sum_{n \geq 0} a_{n+1} z^n \text{ and } B(z) = \sum_{n \geq 0} b_{n+1} z^n.$$

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These two functions satisfy the following Mahler type functional equations

$$A(z) = (1 + z + z^2)A(z^2) \text{ and } B(z) = 2 - (1 + z + z^2)B(z^2), \quad (1.3)$$

see also [7]. For $A(1/b)$ and $B(1/b)$ we have

Theorem 1. *For all integers $b \geq 2$,*

$$\mu \left(A \left(\frac{1}{b} \right), B \left(\frac{1}{b} \right) \right) \leq \frac{8}{5} = 1.6.$$

Moreover, if $a/b \in \mathbb{Q}$ with $\log |a| = \lambda \log b$, where $b \geq 2$, $0 \leq \lambda < 50/77$, then

$$\mu \left(A \left(\frac{a}{b} \right), B \left(\frac{a}{b} \right) \right) \leq \frac{80(1-\lambda)}{(50-77\lambda)}.$$

Now we turn to the following two power series

$$T(z) = \prod_{j=0}^{\infty} (1 - z^{2^j}) \text{ and } M(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2^j}}{\prod_{i=0}^{j-1} (1 - z^{2^i})},$$

in particular $T(z)$ is the generating function of the Thue-Morse sequence on $\{-1, 1\}$. These series are solutions of Mahler type functional equations

$$T(z) = (1 - z)T(z^2) \text{ and } M(z^2) = (z - 1)(M(z) - z), \quad (1.4)$$

and the coefficients of $T(z) = \sum_{j=0}^{\infty} t_j z^j$, $M(z) = \sum_{j=0}^{\infty} m_j z^j$ satisfy, for all $n \geq 1$,

$$\begin{cases} t_0 = 1, \\ t_{2n} = t_n, \\ t_{2n+1} = -t_n, \end{cases} \quad \text{and} \quad \begin{cases} m_0 = 0, \\ m_1 = -m_2 = 1, \\ m_{2n+1} = m_{2n}, \\ m_{2n+2} = m_{2n+1} - m_{n+1}. \end{cases}$$

The numbers $T(\alpha)$ and $M(\alpha)$ are algebraically independent over \mathbb{Q} for any non-zero algebraic number α with $|\alpha| < 1$, see [15, Theorem 4]. For all integers $b \geq 2$, Bugeaud [4] proved that $\mu(T(b^{-1}))$ equals 2 and Väänänen [15] proved that $\mu(M(b^{-1}))$ also equals 2. Thus we have

$$\frac{3}{2} \leq \mu \left(T \left(\frac{1}{b} \right), M \left(\frac{1}{b} \right) \right) \leq \min \left\{ \mu \left(T \left(\frac{1}{b} \right) \right), \mu \left(M \left(\frac{1}{b} \right) \right) \right\} = 2.$$

The following result improves the above upper bound.

Theorem 2. *For all integers $b \geq 2$,*

$$\mu \left(T \left(\frac{1}{b} \right), M \left(\frac{1}{b} \right) \right) \leq \frac{32}{17} = 1.882\dots$$

Moreover, if $a/b \in \mathbb{Q}$ with $\log |a| = \lambda \log b$, $b \geq 2$, $0 \leq \lambda < 1/2$, then

$$\mu \left(T \left(\frac{a}{b} \right), M \left(\frac{a}{b} \right) \right) \leq \frac{32(1-\lambda)}{17-26\lambda}.$$

Our next result studies a special type of Lambert series

$$G_d(z) = \sum_{k=0}^{\infty} \frac{z^{d^k}}{1 - z^{d^k}}$$

and a related function

$$F_d(z) = \sum_{k=0}^{\infty} \frac{z^{d^k}}{1+z^{d^k}}$$

with $d = 3$. These series satisfy the Mahler type functional equations

$$\begin{cases} (z-1)G_d(z) + (1-z)G_d(z^d) + z &= 0, \\ -(1+z)F_d(z) + (1+z)F_d(z^d) + z &= 0. \end{cases} \quad (1.5)$$

Recently Coons [9] proved that $\mu(G_2(1/b)) = \mu(F_2(1/b)) = 2$. The numbers 1, $G_2(1/b)$ and $F_2(1/b)$ are linearly dependent, namely

$$G_2(\alpha) + F_2(\alpha) = \frac{2\alpha}{1-\alpha}$$

for all $|\alpha| < 1$. Thus we get $\mu(G_2(1/b), F_2(1/b)) = 2$. On the other hand, for $d \geq 3$ and algebraic α , $0 < |\alpha| < 1$, two numbers $G_d(\alpha)$ and $F_d(\alpha)$ are known to be algebraically independent, see e.g. [6]. Here we are interested in the simultaneous approximation of $G_3(a/b)$ and $F_3(a/b)$, but we note that our approximation lemma in section 2 could also be used to estimate irrationality exponents of these numbers.

Theorem 3. *For all integers $b \geq 2$,*

$$\mu\left(G_3\left(\frac{1}{b}\right), F_3\left(\frac{1}{b}\right)\right) \leq \frac{36}{19} = 1.894\dots$$

Moreover, if $a/b \in \mathbb{Q}$ with $\log |a| = \lambda \log b$, $b \geq 2$, $0 \leq \lambda < 19/29$, then

$$\mu\left(G_3\left(\frac{a}{b}\right), F_3\left(\frac{a}{b}\right)\right) \leq \frac{36(1-\lambda)}{(19-29\lambda)}.$$

The following result considers the power series

$$S(z) = 1 + z + z^3 + z^4 + z^5 + z^{11} + z^{12} + z^{13} + z^{16} + z^{17} + z^{19} + \dots$$

introduced by Dilcher and Stolarsky [10] and satisfying the Mahler type functional equation

$$S(z^{16}) = -zS(z) + (1+z+z^2)S(z^4) \quad (1.6)$$

of degree 2. This series is connected to Stern polynomials and we see immediately by (1.6) that the coefficients s_k in $S(z) = \sum_{k=0}^{\infty} s_k z^k$ satisfy $s_0 = 1$ and, for all $k \geq 0$,

$$s_{4k} = s_{4k+1} = s_k, \quad s_{4k+2} = 0, \quad s_{4k+3} = \begin{cases} 0, & k \equiv 3 \pmod{4}, \\ s_{k+1}, & \text{otherwise.} \end{cases}$$

In particular, $s_k \in \{0, 1\}$ and, moreover, the indexes k with $s_k = 1$ form a so-called self-generating set, see [10]. In [1] Adamczewski proved that the numbers $S(\alpha)$ and $S(\alpha^4)$ are algebraically independent, if α , $0 < |\alpha| < 1$, is algebraic. Further, by using the gap properties of the series $S(z)$ an upper bound $\mu(S(a/b)) < (5-2\lambda)/(1-2\lambda)$, $b \geq 2$, $0 \leq \lambda < 1/2$, is proved in [8]. Now we prove

Theorem 4. *For all integers $b \geq 2$,*

$$\mu\left(S\left(\frac{1}{b}\right), S\left(\frac{1}{b^4}\right)\right) \leq \frac{516}{253} = 2.039\dots$$

Moreover, if $a/b \in \mathbb{Q}$ with $\log |a| = \lambda \log b$, $b \geq 2$, $0 \leq \lambda < 178/291$, then

$$\mu\left(S\left(\frac{a}{b}\right), S\left(\frac{a^4}{b^4}\right)\right) \leq \frac{516(1-\lambda)}{253-381\lambda}.$$

The above results on simultaneous approximation and Khintchine's transference theorem (see e.g. [14], p. 99) give some information on linear independence exponents $\mu_L(\alpha_1, \alpha_2)$ defined as the supremum of real numbers μ such that the inequality

$$|h_0 + h_1\alpha_1 + h_2\alpha_2| < h^{-\mu}$$

with $h = \max\{|h_i|\}$ has infinitely many solution $(h_0, h_1, h_2) \in \mathbb{Z}^3 \setminus \{\underline{0}\}$. Namely, if $\mu(\alpha_1, \alpha_2) \leq U < 2$, then

$$\mu_L(\alpha_1, \alpha_2) \leq \frac{2}{2 - \mu(\alpha_1, \alpha_2)} - 2 \leq \frac{2}{2 - U} - 2.$$

For example, our Theorems 1-3 imply the following

Corollary 1. *For all integers $b \geq 2$ we have*

$$\mu_L\left(A\left(\frac{1}{b}\right), B\left(\frac{1}{b}\right)\right) \leq 3, \mu_L\left(T\left(\frac{1}{b}\right), M\left(\frac{1}{b}\right)\right) \leq 15, \mu_L\left(G_3\left(\frac{1}{b}\right), F_3\left(\frac{1}{b}\right)\right) \leq 17.$$

However, in a very recent work [17] we obtained better results $\mu_L\left(A\left(\frac{1}{b}\right), B\left(\frac{1}{b}\right)\right) \leq \frac{26}{9} = 2.888\dots$ and $\mu_L\left(G_3\left(\frac{1}{b}\right), F_3\left(\frac{1}{b}\right)\right) \leq \frac{129}{37} = 3,486\dots$ by using another method based on the ideas of Siegel's method.

This paper is organized as follows. In Section 2, basic information about Hermite-Padé approximations and an important approximation lemma are given in a general form for arbitrary number of functions. This lemma gives good simultaneous approximation exponents only if one knows well the asymptotic bounds for linear forms obtained by using Hermite-Padé approximations, and this is generally hard. In the case of two functions studied in our theorems we compute explicitly some approximations and the order (at $z = 0$) of the remainder terms. This computational information is given in [16, Appendix]. As explained at the end of subsection 2.1 we can then produce an infinite sequence of approximations, where the coefficient polynomials and remainders are well controlled and give all we need for the application of the approximation lemma. In Section 3 this application leads to the proof of Theorems 1 to 3 considering Mahler functions of degree one. The proof of Theorem 4, which is more complicated, is presented in Section 4.

2 Preliminaries and approximation lemma

2.1 Hermite-Padé approximation

In this paragraph, we introduce some basic information about the Hermite-Padé approximation to be needed in the following. For general theory, see for example [3, 13].

Let $n \geq 1$ be an integer and $f_1(z), f_2(z), \dots, f_n(z) \in \mathbb{Q}[[z]]$ be formal power series. For given $\mathbf{d} := (d_1, d_2, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$, let $P_1(z), P_2(z), \dots, P_n(z) \in \mathbb{Z}[[z]]$ be non-trivial polynomials such that $\deg P_i \leq d_i$ ($i = 1, 2, \dots, n$) and

$$P_1(z)f_1(z) + P_2(z)f_2(z) + \dots + P_n(z)f_n(z) = R_{\mathbf{d}}(z),$$

where the order of zero at $z = 0$ of the *remainder term* $R_{\mathbf{d}}(z)$, say $\text{ord} R_{\mathbf{d}}(z)$, is at least $s + 1$ where $s := n - 2 + \sum_{i=1}^n d_i$. Such polynomials exist since, using the notations

$$f_j(z) = \sum_{i=0}^{\infty} f_i^{(j)} z^i \text{ and } P_j(z) = \sum_{i=0}^{d_j} p_i^{(j)} z^i,$$

the following system of linear equations

$$\begin{bmatrix} f_0^{(1)} & 0 & \cdots & 0 & f_0^{(n)} & 0 & \cdots & 0 \\ f_1^{(1)} & f_0^{(1)} & \cdots & 0 & f_1^{(n)} & f_0^{(n)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{d_1}^{(1)} & f_{d_1-1}^{(1)} & \cdots & f_0^{(1)} & \cdots & f_{d_n}^{(n)} & f_{d_n-1}^{(n)} & \cdots & f_0^{(n)} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ f_s^{(1)} & f_{s-1}^{(1)} & \cdots & f_{s-d_1}^{(1)} & f_s^{(n)} & f_{s-1}^{(n)} & \cdots & f_{s-d_n}^{(n)} \end{bmatrix} \cdot \begin{bmatrix} p_0^{(1)} \\ \vdots \\ p_{d_1}^{(1)} \\ \vdots \\ p_0^{(n)} \\ \vdots \\ p_{d_n}^{(n)} \end{bmatrix} = \mathbf{0}_{(s+1) \times 1} \quad (2.1)$$

has a non-trivial solution $p_i^{(j)}$. Denote the $(s+1) \times (s+2)$ coefficient matrix in (2.1) by $\Delta_{\mathbf{d}}$. Let

$$X := (p_0^{(1)}, \dots, p_{d_1}^{(1)}, \dots, p_0^{(n)}, \dots, p_{d_n}^{(n)})^T$$

be a non-trivial solution of (2.1) and

$$\delta_j := (f_{s+j}^{(1)}, f_{s+j-1}^{(1)}, \dots, f_{s+j-d_1}^{(1)}, \dots, f_{s+j}^{(n)}, f_{s+j-1}^{(n)}, \dots, f_{s+j-d_n}^{(n)}), j \geq -s,$$

where $f_i^{(j)} = 0$ if $i < 0$. Then

$$R_{\mathbf{d}}(z) = \sum_{j=1}^{\infty} (\delta_j X) \cdot z^{s+j}.$$

The exact order of $R_{\mathbf{d}}(z)$ is $\min\{s+j : \delta_j X \neq 0\}$. Essential for the existence of (d_1, d_2, \dots, d_n) Hermite-Padé approximation of exact order $s+1$ is

$$\delta_1 X \neq 0.$$

This condition is certainly satisfied if the determinant

$$\begin{vmatrix} \Delta_{\mathbf{d}} \\ \delta_1 \end{vmatrix} \neq 0. \quad (2.2)$$

To get explicit simultaneous approximation exponents we need to control well the orders of remainder terms $R_{\mathbf{d}}(z)$. This is the main reason why we consider in the present paper only two Mahler functions, say $f(z)$ and $g(z)$. So we choose above $n=3$, $f_1(z) = f(z)$, $f_2(z) = g(z)$ and $f_3(z) = 1$, take a positive integer k and construct $(d_1, d_2, d_3) := (d_1(k), d_2(k), d_3(k))$ approximation polynomials $A_k(z)$, $B_k(z)$, $C_k(z) \in \mathbb{Z}[z]$, not all zero and of degree $\leq d_1, d_2, d_3$, respectively, such that

$$A_k(z)f(z) + B_k(z)g(z) + C_k(z) = R_k(z),$$

where $\mathfrak{o}(k) := \text{ord} R_k(z) \geq d_1 + d_2 + d_3 + 2$ is computed explicitly, see [16, Appendix]. This will be done for $k = k_1, \dots, k_t$. If we replace above z by z^d ($d=2$ in Theorems 1 and 2, 3 in Theorem 3 and 4 in Theorem 4) and use functional equations, we get a new approximation form. Repeating this we obtain, for each k , an infinite sequence of approximations

$$A_{k,m}(z)f(z) + B_{k,m}(z)g(z) + C_{k,m}(z) = R_{k,m}(z), \quad m = 0, 1, \dots,$$

where the order of $R_{k,m}(z)$ is known. These sequences at $z = a/b$ give then numerical approximation sequences for $f(a/b)$ and $g(a/b)$ used in our proofs.

2.2 Approximation lemma

In the following we shall give our main tool for the proofs, the approximation lemma tailored suitable for the above situation. For this lemma we arrange the pairs (k, m) , where $m \in \mathbb{N}, m \geq m_0 \in \mathbb{N}$, and $k \in \{k_1, \dots, k_t\}$, $k_j \in \mathbb{N}$, $k_1 < k_2 < \dots < k_t$, as follows:

$$(k_1, m_0), \dots, (k_t, m_0), (k_1, m_0 + 1), \dots, (k_t, m_0 + 1), (k_1, m_0 + 2), \dots \quad (2.3)$$

Note that in the case $u = 1$ our lemma gives an upper bound for the irrationality exponent of γ_1 , which is a kind of refinement of a result of Adamczewski and Rivoal [2, Lemma 4.1]. We shall present the lemma in a general form although only the case $u = 2$ is needed in our proofs below.

Lemma 1. *Let $\gamma_1, \dots, \gamma_u$ be real numbers. Assume that for each pair (k, m) in (2.3) there exist a linear form*

$$r(k, m) = h_0 + \sum_{i=1}^u h_i \gamma_i, \quad h_i = h_i(k, m) \in \mathbb{Z},$$

and a positive integer $Q_{k,m}$ such that

$$c_1(k) Q_{k,m} \leq \max_{1 \leq i \leq u} |h_i| \leq c_2(k) Q_{k,m}, \quad (2.4)$$

$$Q_{k_j, m} < Q_{k_{j+1}, m} \leq C_1(\underline{k}) Q_{k_j, m}^{\theta(j)}, \quad j = 1, \dots, t-1, \quad (2.5)$$

$$Q_{k_t, m} < Q_{k_1, m+1} \leq C_1(\underline{k}) Q_{k_t, m}^{\theta(t)}, \quad (2.6)$$

$$c_3(k) Q_{k,m}^{-\alpha(k)} \leq |r(k, m)| \leq c_4(k) Q_{k,m}^{-\beta(k)}, \quad (2.7)$$

where $c_i(k)$ and $C_i(\underline{k})$ are positive constants depending on k and $\underline{k} = \{k_1, \dots, k_t\}$, respectively, and $\theta(j) \geq 1$, $\alpha(k_j) \geq \beta(k_j) > 0$ are constants (all independent of m). Then there exist positive constants $Q_0 = Q_0(\underline{k}, m_0)$ and $C = C(\underline{k}, m_0)$ such that, for all $\frac{p_i}{q} \in \mathbb{Q}$, $q \geq Q_0$,

$$\max_{1 \leq i \leq u} \left| \gamma_i - \frac{p_i}{q} \right| > C q^{-\mu}$$

where $\mu = \max_{1 \leq j \leq t} \theta(j) \frac{\alpha(k_{j+1}) + 1}{\beta(k_j)}$, $\alpha(k_{t+1}) := \alpha(k_1)$.

Proof. For the proof we denote

$$\Delta = \Delta(k, m) := h_0 + \sum_{i=1}^u h_i \frac{p_i}{q}.$$

By defining $\varepsilon_i := \gamma_i - \frac{p_i}{q}$ ($i = 1, \dots, u$) we have

$$\Delta = r(k, m) - \sum_{i=1}^u h_i \varepsilon_i.$$

Assume that, for some (k, m) in (2.3),

$$\max_{1 \leq i \leq u} |\varepsilon_i| < \left(\frac{\min_j c_3(k_j)}{2u \max_j c_2(k_j)} \right) Q_{k,m}^{-\alpha(k)-1} =: C_2(\underline{k}) Q_{k,m}^{-\alpha(k)-1}. \quad (2.8)$$

This implies, by (2.4) and (2.7)

$$\left| \sum_{i=1}^u h_i \varepsilon_i \right| \leq \frac{c_3(k)}{2} Q_{k,m}^{-\alpha(k)} \leq \frac{|r(k, m)|}{2},$$

and therefore

$$0 < |\Delta| \leq \frac{3|r(k, m)|}{2} \leq \frac{3c_4(k)}{2} Q_{k, m}^{-\beta(k)}.$$

Moreover, since $h_i \in \mathbb{Z}$, we have $|\Delta| \geq 1/q$. Thus

$$\log q \geq \beta(k) \log Q_{k, m} - \log \frac{3c_4(k)}{2}. \quad (2.9)$$

Let us define $Q_0 = Q_0(\underline{k}, m_0)$ in such a way that

$$\log Q_0 > \left(\max_{1 \leq j \leq t} \beta(k_j) \right) \log Q_{k_1, m_0} - \left(\min_{1 \leq j \leq t} \log \frac{3c_4(k_j)}{2} \right).$$

We now assume that $q \geq Q_0$ and fix the pair (k_j, m) from (2.3) such that it is the first one satisfying

$$\log q < \beta(k_j) \log Q_{k_j, m} - \log \frac{3c_4(k_j)}{2}, \quad (2.10)$$

by (2.5) and (2.6) such pair exists. From the above choice of Q_0 it follows that $(k_j, m) \neq (k_1, m_0)$. Then (2.8) implying (2.9) cannot hold for the pair (k_j, m) and therefore

$$\max_{1 \leq i \leq u} |\varepsilon_i| \geq C_2(\underline{k}) Q_{k_j, m}^{-\alpha(k_j)-1}. \quad (2.11)$$

From the definition of (k_j, m) it follows that the pair just before it does not satisfy (2.10). For $j > 1$ this pair is (k_{j-1}, m) , and for $j = 1$ it is $(k_t, m-1)$. Thus in the first case

$$\begin{aligned} \log q &\geq \beta(k_{j-1}) \log Q_{k_{j-1}, m} - \log \frac{3c_4(k_{j-1})}{2} \\ &\geq \beta(k_{j-1}) \log Q_{k_{j-1}, m} - \log C_3(\underline{k}), \end{aligned}$$

where $C_3(\underline{k}) = \max_{1 \leq j \leq t} \frac{3}{2} c_4(k_j)$. By (2.5),

$$\log Q_{k_j, m} \leq \log C_1(\underline{k}) + \theta(j-1) \log Q_{k_{j-1}, m},$$

which then implies

$$\begin{aligned} \log q &\geq \frac{\beta(k_{j-1})}{\theta(j-1)} \log Q_{k_j, m} - \log \left(C_1(\underline{k})^{\frac{\beta(k_{j-1})}{\theta(j-1)}} C_3(\underline{k}) \right) \\ &\geq \frac{\beta(k_{j-1})}{\theta(j-1)} \log Q_{k_j, m} - \log C_4(\underline{k}). \end{aligned}$$

Thus

$$q \geq C_4(\underline{k})^{-1} Q_{k_j, m}^{\frac{\beta(k_{j-1})}{\theta(j-1)}}.$$

By (2.11) we now obtain

$$\max_{1 \leq i \leq u} |\varepsilon_i| \geq C_5(\underline{k}) q^{-\mu(j)},$$

where $\mu(j) = \theta(j-1) \frac{\alpha(k_j)+1}{\beta(k_{j-1})}$. Similarly, by using (2.6), we get in the second case

$$\max_{1 \leq i \leq u} |\varepsilon_i| \geq C_6(\underline{k}) q^{-\mu(1)},$$

where $\mu(1) = \theta(t) \frac{\alpha(k_1)+1}{\beta(k_t)}$. These two estimates prove the truth of our lemma. \square

3 Proof of Theorems 1, 2 and 3

In proving Theorems 1, 2 and 3, we shall consider simultaneous approximations of Mahler functions $f(z) = \sum_{j=0}^{\infty} f_j z^j$ and $g(z) = \sum_{j=0}^{\infty} g_j z^j$ converging in the open unit disc and satisfying functional equations of the type

$$\phi_1(z)f(z) + \phi_2(z)f(z^d) + \phi_3(z) = 0, \quad (3.1)$$

$$\psi_1(z)g(z) + \psi_2(z)g(z^d) + \psi_3(z) = 0, \quad (3.2)$$

where $\phi_i, \psi_i \in \mathbb{Z}[z]$ and $d \geq 2$ is a fixed integer. We assume that $|\phi_i(0)| = |\psi_i(0)| = 1$ ($i = 1, 2$). Denote the zeros of ϕ_i, ψ_i ($i = 1, 2$) by

$$\Xi = \{z \in \mathbb{R}, |z| < 1 \mid \phi_i(z) = 0 \text{ or } \psi_i(z) = 0, i = 1, 2\}.$$

Theorems 1, 2 and 3 are obtained from the following general Theorem 5 on simultaneous approximations of the values of $f(z)$ and $g(z)$ at rational points. For it we need some notations. Let $\phi(z) = \text{l.c.m.}(\phi_2(z), \psi_2(z))$ in $\mathbb{Z}[z]$ and define the polynomials $\hat{\phi}_2, \hat{\psi}_2 \in \mathbb{Z}[z]$ by $\phi(z) = \phi_2(z)\hat{\psi}_2(z) = \psi_2(z)\hat{\phi}_2(z)$. Moreover, let v denote the maximal degree of the polynomials $\phi_1\hat{\psi}_2, \hat{\phi}_2\psi_1, \hat{\phi}_2\psi_3, \phi_3\hat{\psi}_2$ and $\phi(z)$. Let $A_k(z), B_k(z)$ and $C_k(z)$ be the $(d_1, d_2, d_3) := (d_1(k), d_2(k), d_3(k))$ simultaneous approximation polynomials of $f(z), g(z)$ and 1, and denote the exact order of the remainder term by $\mathfrak{o}(k)$. Let $\bar{d}(k) := \max\{d_1(k), d_2(k), d_3(k)\}$ and assume that $\bar{d}(k)$ is strictly increasing.

Theorem 5. *Let $f(z)$ and $g(z)$ be the functions given above. Suppose that, for $k \in \{k_1, \dots, k_t\}$ with $k_1 < \dots < k_t$, the (d_1, d_2, d_3) simultaneous approximation polynomials satisfy*

(i) *at least one of $A_k(0)$ and $B_k(0)$ is not zero.*

There exist non-negative integers e_1 and e_2 , depending on (3.1) and (3.2) such that if

(ii) $d \cdot \bar{d}(k_1) + v - \bar{d}(k_t) > (d-1)e_1 + e_2$

and $a/b \in \mathbb{Q}$ with $(a/b)^l \notin \Xi$ ($l \in \mathbb{N}$) and $\log |a| = \lambda \log b$, $b \geq 2$,

$$0 \leq \lambda < \min_{1 \leq j \leq t} \left\{ \frac{\mathfrak{o}(k_j) - (\bar{d}(k_j) - e_1) - \frac{v-e_2}{d-1}}{\mathfrak{o}(k_j)} \right\},$$

then

$$\mu \left(f \left(\frac{a}{b} \right), g \left(\frac{a}{b} \right) \right) \leq \max_{1 \leq j \leq t} \left\{ \frac{(1-\lambda)\mathfrak{o}(k_{j+1})}{(1-\lambda)\mathfrak{o}(k_j) - (\bar{d}(k_j) - e_1) - \frac{v-e_2}{d-1}} \right\},$$

where $\mathfrak{o}(k_{t+1}) := \mathfrak{o}(k_1)d$.

Proof. We start from the type $(d_1, d_2, d_3) := (d_1(k), d_2(k), d_3(k))$ approximation polynomials $A_k(z), B_k(z), C_k(z) \in \mathbb{Z}[z]$ satisfying

$$A_k(z)f(z) + B_k(z)g(z) + C_k(z) = R_k(z).$$

Substituting here z^d for z and applying (3.1) and (3.2), we obtain

$$\begin{aligned} & \hat{\psi}_2(z)\phi_1(z)A_k(z^d)f(z) + \hat{\phi}_2(z)\psi_1(z)B_k(z^d)g(z) \\ & + \hat{\psi}_2(z)\phi_3(z)A_k(z^d) + \hat{\phi}_2(z)\psi_3(z)B_k(z^d) - \phi(z)C_k(z^d) = -\phi(z)R_k(z^d). \end{aligned}$$

Repeating this procedure m times, we have

$$A_{k,m}(z)f(z) + B_{k,m}(z)g(z) + C_{k,m}(z) = R_{k,m}(z), \quad m = 0, 1, \dots, \quad (3.3)$$

where $A_{k,0}(z) = A_k(z)$, $B_{k,0}(z) = B_k(z)$, $C_{k,0}(z) = C_k(z)$, $R_{k,0}(z) = R_k(z)$ and, for $m = 1, 2, \dots$,

$$\begin{aligned} A_{k,m}(z) &= \hat{\psi}_2(z) \phi_1(z) A_{k,m-1}(z^d), \\ B_{k,m}(z) &= \hat{\phi}_2(z) \psi_1(z) B_{k,m-1}(z^d), \\ C_{k,m}(z) &= \hat{\psi}_2(z) \phi_3(z) A_{k,m-1}(z^d) + \hat{\phi}_2(z) \psi_3(z) B_{k,m-1}(z^d) \\ &\quad - \phi(z) C_{k,m-1}(z^d), \\ R_{k,m}(z) &= -\phi(z) R_{k,m-1}(z^d). \end{aligned}$$

Notice that

$$\deg A_{k,m}(z), \deg B_{k,m}(z), \deg C_{k,m}(z) \leq \left(\bar{e}(k) + \frac{\tau}{d-1} \right) \cdot d^m - \frac{\tau}{d-1} \quad (3.4)$$

with $\bar{e}(k) = \bar{d}(k) - e_1$, $\tau = v - e_2$, where e_1 and e_2 are non-negative integers (if $e_1 = e_2 = 0$, then (3.4) certainly holds). Further,

$$\text{ord} R_{k,m}(z) = \mathfrak{o}(k) d^m.$$

Using (3.3), we construct linear forms

$$a_{k,m} f\left(\frac{a}{b}\right) + b_{k,m} g\left(\frac{a}{b}\right) + c_{k,m} = r_{k,m}, \quad m = 0, 1, \dots$$

where

$$a_{k,m} = Q_{k,m} A_{k,m}\left(\frac{a}{b}\right), \quad b_{k,m} = Q_{k,m} B_{k,m}\left(\frac{a}{b}\right), \quad (3.5)$$

$$c_{k,m} = Q_{k,m} C_{k,m}\left(\frac{a}{b}\right), \quad r_{k,m} = Q_{k,m} R_{k,m}\left(\frac{a}{b}\right) \quad (3.6)$$

with $Q_{k,m} = b^{(\bar{e}(k) + \frac{\tau}{d-1}) \cdot d^m - \frac{\tau}{d-1}}$. Here all $a_{k,m}$, $b_{k,m}$ and $c_{k,m}$ are integers.

Suppose $\underline{k} := \{k_1, \dots, k_t\}$ and $k_1 < k_2 < \dots < k_t$. By our assumption, for all $k \in \underline{k}$, we have type (d_1, d_2, d_3) simultaneous approximation polynomials such that at least one of $A_k(0)$ and $B_k(0)$ is not zero.

Now

$$A_{k,m}\left(\frac{a}{b}\right) = A_k\left(\left(\frac{a}{b}\right)^{d^m}\right) \prod_{j=0}^{m-1} \hat{\psi}_2\left(\left(\frac{a}{b}\right)^{d^j}\right) \phi_1\left(\left(\frac{a}{b}\right)^{d^j}\right)$$

implying, by our assumptions on ϕ_i and ψ_i , that

$$c_5(k) \leq \left| A_{k,m}\left(\frac{a}{b}\right) \right| \leq c_6(k)$$

for all $m \geq m_0 = m_0(k)$, if $A_k(0) \neq 0$. $B_{k,m}(a/b)$ can be estimated similarly. Thus the condition (2.4) holds for all $m \geq m_0$. For a given $\delta > 0$ there exists $m_1 = m_1(\delta) > 0$ such that the conditions (2.5) and (2.6) are also satisfied for all $m \geq m_1$ if we choose

$$\theta(j) = \frac{\bar{e}(k_{j+1}) + \frac{\tau}{d-1}}{\bar{e}(k_j) + \frac{\tau}{d-1}} + \delta, \quad j = 1, \dots, t-1, \quad \text{and} \quad \theta(t) = d \cdot \frac{\bar{e}(k_1) + \frac{\tau}{d-1}}{\bar{e}(k_t) + \frac{\tau}{d-1}} + \delta.$$

Note that $\theta(t) > 1$ by the assumption (ii). Moreover

$$R_{k,m}\left(\frac{a}{b}\right) = R_k\left(\left(\frac{a}{b}\right)^{d^m}\right) \prod_{j=0}^{m-1} \phi\left(\left(\frac{a}{b}\right)^{d^j}\right).$$

Since $f(z)$ and $g(z)$ converge in the open unit disc we may choose m_0 above in such a way that

$$c_7(k) \left(\frac{|a|}{b}\right)^{\mathfrak{o}(k) d^m} \leq \left| R_{k,m}\left(\frac{a}{b}\right) \right| \leq c_8(k) \left(\frac{|a|}{b}\right)^{\mathfrak{o}(k) d^m}$$

for all $m \geq m_0$. Therefore (2.7) also holds for all $m \geq m_2 = m_2(\delta)$ with

$$\alpha(k) - \delta = \beta(k) = \frac{(1-\lambda)\mathfrak{o}(k)}{\bar{e}(k) + \frac{\tau}{d-1}} - 1.$$

For $j = 1, \dots, t-1$,

$$\theta(j) \frac{\alpha(k_{j+1}) + 1}{\beta(k_j)} \leq \frac{(1+\delta) \left[(1-\lambda)\mathfrak{o}(k_{j+1}) + \delta(\bar{e}(k_{j+1}) + \frac{\tau}{d-1}) \right]}{(1-\lambda)\mathfrak{o}(k_j) - \bar{e}(k_j) - \frac{\tau}{d-1}},$$

and

$$\theta(t) \frac{\alpha(k_1) + 1}{\beta(k_t)} \leq \frac{(1+\delta)d \left[(1-\lambda)\mathfrak{o}(k_1) + \delta(\bar{e}(k_1) + \frac{\tau}{d-1}) \right]}{(1-\lambda)\mathfrak{o}(k_t) - \bar{e}(k_t) - \frac{\tau}{d-1}}.$$

Applying Lemma 1, we are done, since $\delta > 0$ can be chosen arbitrarily small. \square

Note that if we use above only one value $k = k_1$, then the upper bound for μ in Theorem 5 is greater than d . Therefore, to get sharp approximation exponents we necessarily need to use several values of k .

Now we shall prove Theorems 1, 2 and 3.

Proof of Theorem 1. We choose $d = 2$, $f(z) = A(z)$ and $g(z) = B(z)$, and apply Theorem 5. By (1.3), $\phi_1(z) = 1$, $\phi_2(z) = -(1+z+z^2)$, $\phi_3(z) = 0$, $\psi_1(z) = 1$, $\psi_2(z) = 1+z+z^2$ and $\psi_3(z) = -2$. Thus the conditions given before Theorem 5 are satisfied. Further, $\phi(z) = 1+z+z^2$ and $\hat{\phi}_2(z) = 1$, $\hat{\psi}_2(z) = -1$, $\nu = 2$.

For all k , $7 \leq k \leq 51$, the determinant

$$\left| \frac{\Delta_{k,k+1,k-1}}{\delta_1} \right| \neq 0,$$

where we use the notations of subsection 2.1 (see [16, Appendix A]). This implies the condition (i) of Theorem 5. Moreover, as noted in subsection 2.1, we have $\mathfrak{o}(k) = \text{ord}R_k(z) = 3k+2$. Now we may take $\bar{e}(k) = k$ and $\tau = 2$ in (3.4), so $e_1 = 1$ and $e_2 = 0$. Assume now that k_1 , $8 \leq k_1 \leq 25$, is given and choose $k_2 = k_1 + 1$, $k_3 = k_1 + 2, \dots, k_t = 2k_1 + 1$ ($t = k_1 + 2$). Then also the condition (ii) of Theorem 5 is satisfied. By this Theorem

$$\begin{aligned} \mu \left(A \left(\frac{a}{b} \right), B \left(\frac{a}{b} \right) \right) &\leq \max \left\{ \max_{1 \leq j \leq t-1} \frac{(1-\lambda)(3k_j+5)}{2k_j - \lambda(3k_j+2)}, \frac{(1-\lambda)(3k_t+1)}{2k_t - \lambda(3k_t+2)} \right\} \\ &= \frac{(1-\lambda)(3k_1+5)}{2k_1 - \lambda(3k_1+2)}, \end{aligned}$$

if $\lambda < \frac{2k_1}{3k_1+2}$. The choice $k_1 = 25$ gives Theorem 1. \square

Remark 1. By the discussion in subsection 2.1 the above proof would give $\mu(A(1/b), B(1/b)) = 3/2$, if one could prove that the determinant

$$\left| \frac{\Delta_{k,k+1,k-1}}{\delta_1} \right| \neq 0, \tag{3.7}$$

for all $k \geq k_0$. However, the determinants (3.7) are more complicated than the Hankel determinants of one function $A(z)$ or $B(z)$ used in the consideration of $\mu(A(1/b))$ and $\mu(B(1/b))$. For the research of such Hankel determinants see [5, 9, 11, 12, 15, 18] and references there in.

Proof of Theorem 2. In this proof we use Theorem 5 with $d = 2$, $f(z) = T(z)$ and $g(z) = M(z)$. Now $\phi_1(z) = 1$, $\phi_2(z) = z-1$, $\phi_3(z) = 0$, $\psi_1(z) = z-1$, $\psi_2(z) = 1$, $\psi_3(z) = z(1-z)$. Thus $\phi(z) = 1-z$, $\hat{\phi}_2(z) = 1-z$ and $\hat{\psi}_2(z) = -1$, which gives $\nu = 3$.

We construct $(k, k, k+1)$ approximations with 9 values $k = k_j$ given in the following table:

j	1	2	3	4	5	6	7	8	9
k_j	8	9	10	11	12	13	14	15	16
$\mathfrak{o}(k_j)$	32	32	33	36	39	42	45	48	52

The polynomials A_k, B_k and also $\mathfrak{o}(k) := \text{ord}R_k(z)$ are given in [16, Appendix B], where we also see that all $A_k(0) \neq 0$. Again, in this special case we have in (3.4) $\bar{e}(k) = k + 1$, $\tau = 1$ giving $e_1 = 0$, $e_2 = 2$. Thus the condition (ii) of Theorem 5 holds, and we obtain

$$\begin{aligned} \mu\left(T\left(\frac{a}{b}\right), M\left(\frac{a}{b}\right)\right) &\leq \max_{1 \leq j \leq 9} \frac{(1-\lambda)\mathfrak{o}(k_{j+1})}{\mathfrak{o}(k_j) - (k_j + 2) - \lambda\mathfrak{o}(k_j)} \\ &= \frac{2(1-\lambda)\mathfrak{o}(k_1)}{\mathfrak{o}(k_9) - (k_9 + 2) - \lambda\mathfrak{o}(k_9)} = \frac{32(1-\lambda)}{17-26\lambda}, \end{aligned}$$

if $\lambda < 1/2$. This proves Theorem 2. □

In a similar way, we can prove Theorem 3.

Proof of Theorem 3. In this case $d = 3$, $f(z) = G_3(z)$ and $g(z) = F_3(z)$. By the functional equations (1.5), $\phi_1(z) = z - 1$, $\phi_2(z) = 1 - z$, $\phi_3(z) = z$, $\psi_1(z) = -(1 + z)$, $\psi_2(z) = 1 + z$, $\psi_3(z) = z$. Thus $\phi(z) = 1 - z^2$, $\hat{\phi}_2(z) = 1 - z$ and $\hat{\psi}_2(z) = 1 + z$. We have $v = 2$.

We shall use (k, k, k) approximations with 6 values of k , which are given in [16, Appendix C] and satisfy (i) in Theorem 5. The important parameters are here

j	1	2	3	4	5	6
k_j	9	10	13	18	22	26
$\mathfrak{o}(k_j)$	29	36	45	56	70	80

In (3.4) we now have $\bar{e}(k) = k$, $\tau = 2$. So $e_1 = e_2 = 0$, and again the condition (ii) is satisfied. By Theorem 5 it follows that

$$\begin{aligned} \mu\left(G_3\left(\frac{a}{b}\right), F_3\left(\frac{a}{b}\right)\right) &\leq \max_{1 \leq j \leq 6} \frac{(1-\lambda)\mathfrak{o}(k_{j+1})}{(1-\lambda)\mathfrak{o}(k_j) - \bar{e}(k_j) - \frac{\tau}{d-1}} \\ &= \frac{(1-\lambda)\mathfrak{o}(k_2)}{(1-\lambda)\mathfrak{o}(k_1) - \bar{e}(k_1) - 1} = \frac{36(1-\lambda)}{19-29\lambda}, \end{aligned}$$

if $\lambda < 19/29$, which proves Theorem 3. □

4 Proof of Theorem 4

The function $S(z)$ satisfies the functional equation

$$S(z^{16}) = -zS(z) + (1 + z + z^2)S(z^4).$$

Therefore, starting from $(k, k, k - 1)$ Hermite-Padé approximation

$$A_k(z)S(z) + B_k(z)S(z^4) + C_k(z) = R_k(z)$$

we obtain an infinite sequence of approximations

$$A_{k,m}(z)S(z) + B_{k,m}(z)S(z^4) + C_{k,m}(z) = R_{k,m}(z), \quad m = 0, 1, \dots \quad (4.1)$$

where $A_{k,0}(z) = A_k(z)$, $B_{k,0}(z) = B_k(z)$, $C_{k,0}(z) = C_k(z)$, $R_{k,0}(z) = R_k(z)$, and for $m \geq 0$,

$$\begin{aligned} A_{k,m+1}(z) &= -zB_{k,m}(z^4), & B_{k,m+1}(z) &= (1+z+z^2)B_{k,m}(z^4) + A_{k,m}(z^4), \\ C_{k,m+1}(z) &= C_{k,m}(z^4), & R_{k,m+1}(z) &= R_{k,m}(z^4). \end{aligned} \quad (4.2)$$

By the above recursions (4.2), $\deg A_{k,m}$ and $\deg B_{k,m}$ are at most $k \cdot 4^m + 2(1 + 4 + \dots + 4^{m-1}) = k \cdot 4^m + 2(4^m - 1)/3$, and $\deg C_{k,m} \leq k \cdot 4^m$.

We need to estimate the absolute values of $A_{k,m}(z)$ and $B_{k,m}(z)$ at $z = a/b$, $|a| < b$. For these considerations we assume that $A_k(0) = 0$, $B_k(0) \neq 0$. Then there exists $m_0 = m_0(k, a/b)$ such that

$$\left| A_k(z^{4^m}) \right| \leq \left| z^{2 \cdot 4^{m-1}} \right| \cdot \left| B_k(z^{4^m}) \right|, \quad \frac{|B_k(0)|}{2} \leq \left| B_k(z^{4^m}) \right| \leq \frac{3|B_k(0)|}{2} \quad (4.3)$$

for all $m \geq m_0$ and $-|a|/b \leq z \leq |a|/b$. Then, by (4.2) and the first inequality in (4.3),

$$\begin{aligned} \left| B_{k,1}(z^{4^{m-1}}) \right| &\leq (1 + z^{4^{m-1}} + z^{2 \cdot 4^{m-1}}) \left| B_k(z^{4^m}) \right| + \left| A_k(z^{4^m}) \right| \leq (1 + z^{4^{m-1}} + 2z^{2 \cdot 4^{m-1}}) \left| B_k(z^{4^m}) \right|, \\ \left| B_{k,1}(z^{4^{m-1}}) \right| &\geq (1 + z^{4^{m-1}} + z^{2 \cdot 4^{m-1}}) \left| B_k(z^{4^m}) \right| - \left| A_k(z^{4^m}) \right| \geq (1 + z^{4^{m-1}}) \left| B_k(z^{4^m}) \right|, \\ \left| A_{k,1}(z^{4^{m-1}}) \right| &= \left| z^{4^{m-1}} \right| \cdot \left| B_k(z^{4^m}) \right| \leq \frac{z^{4^{m-1}}}{1 + z^{4^{m-1}}} \left| B_{k,1}(z^{4^{m-1}}) \right|. \end{aligned}$$

Repeating this we get

$$\begin{aligned} \left| B_{k,2}(z^{4^{m-2}}) \right| &\leq (1 + z^{4^{m-2}} + 2z^{2 \cdot 4^{m-2}}) \left| B_{k,1}(z^{4^{m-1}}) \right|, \\ \left| B_{k,2}(z^{4^{m-2}}) \right| &\geq (1 + z^{4^{m-2}}) \left| B_{k,1}(z^{4^{m-1}}) \right|, \\ \left| A_{k,2}(z^{4^{m-2}}) \right| &\leq \left| z^{4^{m-2}} \right| \cdot \left| B_{k,1}(z^{4^{m-1}}) \right| \leq \frac{z^{4^{m-2}}}{1 + z^{4^{m-2}}} \left| B_{k,2}(z^{4^{m-2}}) \right|. \end{aligned}$$

After m steps we have

$$\begin{aligned} |B_{k,m}(z)| &\leq \left| B_k(z^{4^m}) \right| \prod_{l=0}^{m-1} (1 + z^{4^l} + 2z^{2 \cdot 4^l}), \\ |B_{k,m}(z)| &\geq \left| B_k(z^{4^m}) \right| \prod_{l=0}^{m-1} (1 + z^{4^l}), \\ |A_{k,m}(z)| &\leq \frac{|z|}{1 + z} |B_{k,m}(z)|. \end{aligned}$$

By (4.3) we therefore obtain, for all $m \geq m_0$,

$$\begin{aligned} \left| B_{k,m} \left(\frac{a}{b} \right) \right| &\leq \frac{3|B_k(0)|}{2} \prod_{l=0}^{\infty} \left(1 + \left(\frac{|a|}{b} \right)^{4^l} + 2 \left(\frac{|a|}{b} \right)^{2 \cdot 4^l} \right) =: \hat{c}_1(k), \\ \left| B_{k,m} \left(\frac{a}{b} \right) \right| &\geq \frac{|B_k(0)|}{2} \prod_{l=0}^{\infty} \left(1 + \left(\frac{|a|}{b} \right)^{4^l} \right) =: \hat{c}_2(k), \\ \left| A_{k,m} \left(\frac{a}{b} \right) \right| &\leq \frac{|a|}{b - |a|} \left| B_{k,m} \left(\frac{a}{b} \right) \right|. \end{aligned}$$

In our proof we shall use the following values.

j	1	2	3	4	5	6	7	8	9	10
k_j	16	21	27	32	37	42	47	52	57	63
$\mathfrak{o}(k_j)$	64	64	82	108	112	127	172	172	172	190

In all these cases $A_{k_j}(0) = 0$ and $B_{k_j}(0) \neq 0$, see [16, Appendix D]. We now construct linear forms $r(k_j, m)$, by multiplying (4.1), where $z = a/b$ and $k = k_j$, with

$$Q_{k_j, m} = b^{(k_j + \frac{2}{3}) \cdot 4^m - \frac{2}{3}}.$$

For a given $\delta > 0$ there exists $m_1 > 0$ such that

$$\begin{aligned} \frac{(k_{j+1} + \frac{2}{3}) \cdot 4^m - \frac{2}{3}}{(k_j + \frac{2}{3}) \cdot 4^m - \frac{2}{3}} &\leq \frac{k_{j+1} + \frac{2}{3}}{k_j + \frac{2}{3}} + \delta, \quad j = 1, 2, \dots, 9 \\ \frac{(k_1 + \frac{2}{3}) \cdot 4^{m+1} - \frac{2}{3}}{(k_{10} + \frac{2}{3}) \cdot 4^m - \frac{2}{3}} &< \frac{4(k_1 + \frac{2}{3})}{k_{10} + \frac{2}{3}} + \delta, \end{aligned}$$

for all $m \geq m_1$. By the above consideration the conditions (2.4), (2.5) and (2.6) of Lemma 1 are satisfied for all $m \geq \max\{m_0, m_1\}$ if we choose

$$\theta(j) = \frac{k_{j+1} + \frac{2}{3}}{k_j + \frac{2}{3}} + \delta, \quad j = 1, 2, \dots, 9, \text{ and } \theta(10) = \frac{4(k_1 + \frac{2}{3})}{k_{10} + \frac{2}{3}} + \delta.$$

Moreover, we may choose m_1 above in such a way that also (2.7) is satisfied with

$$\alpha(k_j) - \delta = \beta(k_j) = \frac{(1 - \lambda)\mathfrak{o}(k_j) - k_j - \frac{2}{3}}{k_j + \frac{2}{3}}, \quad j = 1, 2, \dots, 10$$

for all $m \geq m_1$.

Since we may choose δ above arbitrarily small and

$$\begin{aligned} &\max \left\{ \max_{1 \leq j \leq 9} \frac{(1 - \lambda)\mathfrak{o}(k_{j+1})}{(1 - \lambda)\mathfrak{o}(k_j) - (k_j + \frac{2}{3})}, \frac{4(1 - \lambda)\mathfrak{o}(k_1)}{(1 - \lambda)\mathfrak{o}(k_{10}) - (k_{10} + \frac{2}{3})} \right\} \\ &= \frac{(1 - \lambda)\mathfrak{o}(k_7)}{(1 - \lambda)\mathfrak{o}(k_6) - (k_6 + \frac{2}{3})} = \frac{172(1 - \lambda)}{127(1 - \lambda) - (42 + \frac{2}{3})}, \end{aligned}$$

if $\lambda < 178/291 = 0.611\dots$, Theorem 4 follows from Lemma 1. □

Appendices

A Values of determinant (3.7) for $7 \leq k \leq 51$

For $7 \leq k \leq 51$, values of determinant (3.7) (mod 49) are: 37, 13, 3, 10, 6, 22, 24, 47, 13, 19, 46, 47, 27, 2, 44, 28, 28, 12, 20, 34, 30, 5, 5, 46, 2, 39, 35, 44, 14, 4, 12, 47, 10, 2, 31, 36, 13, 16, 43, 46, 7, 5, 21, 15, 21.

B Approximation polynomials in Theorem 2

Here we list the approximation polynomials $A_k(z)$ and $B_k(z)$ and the order $\mathfrak{o}(k)$ of $R_k(z)$.

k	$\mathfrak{o}(k)$	$A_k(z)$
		$B_k(z)$
8	32	$z^8 + 2z^4 + 1$
		$-2z^6 + 4z^5 + 2z^4 - 8z^3 + 2z^2 + 4z - 2$

k	$\mathfrak{o}(k)$	$A_k(z)$
		$B_k(z)$
9	32	$5z^9 - z^8 + 2z^5 - 10z^4 - 4z^3 - 4z^2 + z - 5$
		$16z^9 - 16z^8 - 10z^7 + 22z^6 - 26z^5 - 10z^4 + 34z^3 + 2z^2 - 6z - 6$
10	33	$z^9 + 4z^5 + 2z^4 + z^3 + z^2 + 2z + 1$
		$-4z^9 + 4z^8 - 2z^7 + 4z^6 + 10z^5 - 16z^4 - 2z^3 + 8z^2 - 4z + 2$
11	36	$2z^{10} + 2z^6 + z^4 + 2z^2 + 1$
		$4z^{10} - 8z^9 + 8z^7 - 4z^6 + 2z^2 - 4z + 2$
12	39	$2z^{12} + 2z^{11} - 2z^{10} - 2z^8 - z^6 + z^3 + z - 1$
		$12z^{12} - 16z^{11} - 8z^{10} + 16z^9 - 12z^8 + 12z^7 + 4z^6 - 12z^5 + 10z^4 - 8z^3 - 2z^2 + 6z - 2$
13	42	$2z^{13} + 2z^{12} + 4z^{11} + 4z^{10} + 4z^9 + 4z^8 + 2z^7 + 2z^6 + 3z^5 + 3z^4 + 3z^3 + 3z^2 + 2z + 2$
		$8z^{11} - 8z^{10} - 4z^9 + 4z^8 - 12z^7 + 12z^6 + 8z^5 - 8z^4 + 6z^3 - 6z^2 - 4z + 4$
14	45	$10z^{14} + 16z^{13} + 20z^{12} + 24z^{11} + 20z^{10} + 20z^9 + 11z^8 + 8z^7 + 13z^6 + 18z^5 + 18z^4 + 20z^3 + 14z^2 + 12z + 4$
		$12z^{14} - 8z^{13} + 20z^{12} - 8z^{11} - 44z^{10} + 16z^9 - 28z^8 + 16z^7 + 74z^6 - 20z^5 - 8z^4 - 8z^3 - 36z^2 + 8z + 16$
15	48	$z^{15} + z^{14} - z^{11} - z^{10} - z^9 - z^8 + z^7 + z^6 - z^3 - z^2 - z - 1$
		$2z^{13} - 2z^{12} - 6z^{11} + 6z^{10} + 14z^7 - 14z^6 - 8z^5 + 8z^4 - 8z^3 + 8z^2 + 6z - 6$
16	52	$z^{16} + 6z^{15} + 11z^{14} + 16z^{13} + 17z^{12} + 18z^{11} + 15z^{10} + 12z^9 + 7z^8 + 10z^7 + 13z^6 + 16z^5 + 15z^4 + 14z^3 + 10z^2 + 6z + 1$
		$4z^{16} + 2z^{14} + 8z^{13} - 4z^{12} - 16z^{11} - 10z^{10} - 8z^9 + 8z^8 + 32z^7 + 14z^6 - 8z^5 - 12z^4 - 16z^3 - 6z^2 + 8z + 6$

C Approximation polynomials in Theorem 3

k	$\mathfrak{o}(k)$	$A_k(z)$
		$B_k(z)$
9	29	$-179z^9 + z^8 + z^7 + 14z^6 + z^5 + z^4 + 14z^3 + z^2 + z + 141$
		$z^9 + z^8 - z^7 + 14z^6 - z^5 + z^4 - 14z^3 + z^2 - z - 37$
10	36	$z^{10} + z^9 - z - 1$
		$z^{10} + z^9 + z + 1$
13	45	$26z^{13} - 2z^{11} - 2z^9 - 4z^7 - 15z^4 + z^2 - 4z + 1$
		$-26z^{13} + 2z^{11} + 2z^9 + 4z^7 - 15z^4 + z^2 + 4z + 1$
18	56	$142z^{18} - z^{16} - z^{14} - 14z^{12} - z^{10} - 52z^9 - z^8 - 14z^6 - z^4 - z^2 - 52$
		$-142z^{18} + z^{16} + z^{14} + 14z^{12} + z^{10} - 52z^9 + z^8 + 14z^6 + z^4 + z^2 + 52$
22	70	$38z^{22} - 3z^{20} - z^{19} - 3z^{18} - 4z^{16} - 14z^{13} + z^{11} - z^{10} + z^9 - 14z^4 + z^2 - z + 1$
		$-38z^{22} + 3z^{20} - z^{19} + 3z^{18} + 4z^{16} - 14z^{13} + z^{11} + z^{10} + z^9 + 14z^4 - z^2 - z - 1$
26	80	$-379z^{26} + z^{25} + z^{24} + 11z^{23} + z^{22} + z^{21} + 11z^{20} + z^{19} + z^{18} + 141z^{17} + z^{16} + z^{15} + 11z^{14} + z^{13} + z^{12} + 11z^{11} + z^{10} + z^9 + 141z^8 + z^7 + z^6 + 11z^5 + z^4 + z^3 + 11z^2 + z + 1$
		$379z^{26} + z^{25} - z^{24} + 11z^{23} - z^{22} + z^{21} - 11z^{20} + z^{19} - z^{18} + 141z^{17} - z^{16} + z^{15} - 11z^{14} + z^{13} - z^{12} + 11z^{11} - z^{10} + z^9 - 141z^8 + z^7 - z^6 + 11z^5 - z^4 + z^3 - 11z^2 + z - 1$

D Approximation polynomials in Theorem 4

k	$\mathfrak{o}(k)$	$A_k(z)$ $B_k(z)$
16	64	$z^{15} - z^{13} + z^{11} + z^7 + z$ $-z^{16} - z^{15} + z^{13} - z^{11} - 2z^8 - z^7 - z^6 + z^4 - z^2 - z - 1$
21	64	$z^{20} + 2z^{16} + z^{14} - z^{13} + z^{12} - z^4 - z^2 + z$ $-z^{21} - z^{20} - z^{19} - 2z^{17} - 2z^{16} - 2z^{15} - z^{13} + z^9 - z^8 - z^7 + 2z^4 + 2z^3 - 1$
27	82	$z^{27} - z^{25} + z^{23} - 2z^{21} + z^{19} - 2z^9 - 2z^5 - z$ $-z^{27} + z^{25} - z^{24} - z^{23} + 2z^{22} + 2z^{21} - z^{19} - z^{18} + 2z^{10} + 2z^9 + 2z^8 + 2z^6 + 2z^5 + z^2 + z + 1$
32	108	$z^{29} + 2z^{25} + z^{21} + 2z^{13} + 4z^9 + 4z^5 + 2z$ $-z^{32} - z^{30} - z^{29} - z^{28} - 2z^{26} - 2z^{25} - z^{24} - z^{22} - z^{21} - 2z^{14} - 2z^{13} - 2z^{12} - 4z^{10} - 4z^9 - 2z^8 - 4z^6 - 4z^5 - 2z^4 - 2z^2 - 2z - 1$
37	112	$z^{37} + z^{33} - 2z^{31} - 4z^{30} - z^{29} + 2z^{25} + z^{23} + 2z^{22} + 2z^{21} - 2z^{19} - 4z^{18} - 2z^{17} - 4z^{15} - 8z^{14} - 2z^{13} - 2z^{11} - 4z^{10} + 2z^9 + 3z^5 + z^3 + 2z^2 + 2z$ $z^{37} - z^{34} - z^{33} + z^{32} + 6z^{31} + 6z^{30} + 3z^{29} - 3z^{26} - 4z^{25} - 3z^{24} - 3z^{23} - 4z^{22} - 2z^{21} + 2z^{20} + 6z^{19} + 8z^{18} + 6z^{17} + 6z^{16} + 12z^{15} + 12z^{14} + 6z^{13} + 2z^{12} + 6z^{11} + 2z^{10} - 2z^9 - 2z^8 - 2z^6 - z^5 - z^4 - 3z^3 - 5z^2 - 4z - 2$
42	127	$z^{42} + z^{41} - z^{40} + z^{38} + z^{37} - 2z^{36} + z^{34} + z^{33} + z^{32} + z^{26} + z^{25} - 3z^{24} + z^{22} + z^{21} - 2z^{20} + z^{10} + z^9 + z^6 + z^5 - z^4 + z^2 + z$ $-2z^{42} - z^{41} - z^{39} - 2z^{38} + z^{37} + 2z^{36} - 2z^{34} - 3z^{33} - 2z^{32} - z^{31} - 2z^{26} + z^{25} + 2z^{24} + z^{23} - 2z^{22} + z^{21} + 2z^{20} - 2z^{10} - 2z^9 - z^8 - 2z^7 - 2z^6 + z^4 - 2z^2 - 2z - 1$
47	172	$-z^{46} + z^{45} - z^{42} + 2z^{41} - z^{38} + 2z^{37} + z^{33} - z^{30} + z^{29} - z^{26} + 2z^{25} - z^{17} - z^{14} - z^{10} + 2z^9 - z^6 + z^5$ $z^{47} - z^{44} + z^{43} - z^{42} - 2z^{41} - z^{40} + z^{39} - z^{38} - z^{37} - z^{36} - z^{34} - z^{33} - z^{32} + z^{31} - z^{28} + z^{27} - z^{26} - 2z^{25} - z^{24} + z^{20} + z^{18} + z^{17} + z^{15} + z^{14} + z^{13} + z^{11} - z^{10} - 2z^9 - z^8 + z^7 - 1$
52	172	$-z^{51} - z^{48} - 2z^{44} - z^{43} + z^{41} - 2z^{40} + z^{37} - z^{36} - 2z^{35} + z^{33} - z^{32} + z^{25} - z^{24} - z^{17} - 3z^{16} + z^{15} - z^{13} - z^{11} + z^9 - 2z^8 + z^7 - 3z^4$ $z^{52} + z^{51} + z^{50} + z^{49} + z^{48} + z^{47} - z^{46} + 2z^{45} + 3z^{44} + 2z^{43} + z^{42} + z^{41} + z^{40} + z^{39} - 2z^{38} + 3z^{36} + 3z^{35} + z^{34} + z^{31} - z^{30} - z^{27} + 2z^{23} - z^{22} + z^{20} - 2z^{19} + 2z^{18} + 4z^{17} + 2z^{16} + 2z^{15} - z^{14} + z^{13} + 2z^{12} + z^{10} + z^9 + 2z^7 - 2z^6 + 3z^5 + 4z^4 + z^3 + z^2 - 1$
57	172	$-z^{53} - 2z^{49} + 7z^{17} + 5z^{13} - 4z^9 + 7z^5 + 5z$ $z^{54} + z^{53} + z^{52} + 2z^{50} + 2z^{49} + z^{48} - z^{44} + 2z^{40} - z^{36} - z^{32} + 2z^{28} - 2z^{24} - z^{20} - 7z^{18} - 7z^{17} - 2z^{16} - 5z^{14} - 5z^{13} - 3z^{12} + 4z^{10} + 4z^9 + 2z^8 - 7z^6 - 7z^5 - 8z^4 - 5z^2 - 5z + 1$
63	190	$-z^{62} + z^{58} + 2z^{57} - 2z^{55} + z^{54} + 2z^{53} - 2z^{51} + 3z^{50} + 3z^{49} - 3z^{47} + z^{42} + z^{41} - z^{39} + z^{38} + z^{21} - z^{19} + 3z^{18} + 3z^{17} - 3z^{15} + z^{14} - z^9 - z^6 - 2z^5 + z^3 - z^2 - z$ $z^{63} + z^{62} + z^{61} - z^{59} - 3z^{58} - 4z^{57} + z^{55} - z^{54} - z^{53} + 2z^{52} - z^{51} - 6z^{50} - 6z^{49} + 3z^{47} + 3z^{46} + z^{45} + z^{44} - z^{43} - 3z^{42} - 3z^{41} - z^{40} + z^{38} + z^{36} - z^{34} + z^{29} + z^{28} - z^{26} - z^{25} - z^{24} + z^{20} - 2z^{19} - 6z^{18} - 6z^{17} + 2z^{15} + 2z^{14} + z^{12} + z^9 + z^7 + 4z^6 + 2z^5 + z^2 + 2z + 1$

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